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# Two-loop splitting functions in QCD*

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Abstract: We present the universal two-loop splitting functions that describe the limits of two-loop $n$-point amplitudes of massless particles when two of the momenta are collinear. To derive the splitting amplitudes, we take the collinear limits of explicit two-loop four-point helicity amplitudes computed in the 't Hooft-Veltman scheme. The $g \to gg$ splitting amplitude has recently been computed using the unitarity sewing method and we find complete agreement with the results of ref. [1]. The two-loop $q \to qg$ and $g \to q\bar{q}$ splitting functions are new results. We also provide an expression for the two-loop soft splitting function.

Keywords: QCD, NLO Computations, Jets.

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# 1. Introduction

The singular behaviour of QCD amplitudes is an important ingredient in understanding the perturbative structure of quantum field theories. In general, when one or more final state particles are either soft or collinear, the amplitudes factorise into the product of an amplitude depending on the remaining hard partons in the process (including any hard partons constructed from an ensemble of unresolved partons) and a factor that contains all of the singularities due to the unresolved particles. One of the best known examples of this type of factorisation is the limit of tree amplitudes when two particles are collinear. This factorisation is universal and can be generalised to any number of loops [2].

Such factorisation properties play a dual role in developing higher order perturbative predictions for observable quantities. On the one hand, a detailed knowledge of the structure of unresolved emission enables phase space integrations to be organised such that the infrared singularities due to soft or collinear emission can be analytically extracted [3]–[5]. On the other, the collinear limit plays a crucial role in the unitarity-based method for loop calculations [6]–[9].

For next-to-leading order (NLO) predictions one of the key ingedients is knowledge of the the single unresolved configurations where a gluon is soft or where two particles are collinear. The universal behaviour of tree amplitudes in the single soft and collinear limits is well known and has been extensively discussed in the literature (see for example refs. [10]–[14]). Antenna factors that describe both limits simultaneously have also been derived [15] and subsequently employed in the construction of parton level NLO predictions for $e^+e^- \to$ 4 jets [16]. At next-to-next-to-leading order (NNLO) one encounters, for example, tree amplitudes with double unresolved configurations where two gluons are soft [17, 18] or when three particles are collinear [19]–[21] as well as the single unresolved limits of one-loop amplitudes [6] and [22]–[26]. Currently, there is significant effort to extract the infrared singularities from such processes [27]–[36] and to combine them with the recently computed two-loop amplitudes for parton-parton scattering [37]–[46], massless Bhabha scattering [47], light-by-light scattering [48, 49] as well as the gluonic production of photons [50] and $\gamma^* \to 3$ partons [51]–[53] to provide NNLO estimates of the dominant QCD processes in electron-positron annihilation and hadron-hadron collisions.

One of the recent highlights in perturbative QCD is the succesful computation of the three-loop contributions to the Altarelli-Parisi kernels that determine the scale evolution of parton densities. These splitting functions have recently been computed in a very impressive series of papers by Moch, Vermaseren and Vogt [54]–[56] using traditional methods and superseding previous approximations based on a limited number of moments [57]–[59]. However, an alternative method using the collinear factorisation properties of QCD amplitudes has been proposed by Kosower and Uwer [60]. At this order one encounters contributions from tree amplitudes with four collinear particles [61], one-loop graphs with three collinear particles [62] and two-loop graphs with two collinear particle [1].

The two-loop splitting functions describing the time-like splitting of one particle into two collinear massless particles are the subject of this paper. They have been studied by Bern, Dixon and Kosower [1] who used the unitarity method of sewing together tree (and one-loop) amplitudes to construct two-loop amplitudes in the limit where two gluons are collinear both in the case of QCD and $N = 1$ and $N = 4$ super Yang-Mills. In this approach many of the problems associated with using the light-cone gauge in traditional Feynman diagram computations of the splitting functions are avoided. In adition, they have checked that the infrared singularities of the splitting function agree with the general formula of Catani [63, 64] and provided an ansatz for the form of the non-trivial colour structures that appear at $\mathcal{O}(1/\epsilon)$ therein.



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In this paper, we exploit the universal factorisation properties of QCD to extract the two-loop splitting functions for all parton combinations by taking the collinear limits of existing two-loop calculations of four-point helicity amplitudes [65]–[69] and relating them to the two-loop amplitudes for vertex graphs. For example, the quark-gluon splitting function is obtained through to $\mathcal{O}(\epsilon^0)$ by taking the collinear limit of the helicity amplitudes for $\gamma^* \to q\bar{q}g$ computed in the 't Hooft-Veltman scheme [70] and relating it to the two-loop amplitudes for $\gamma^* \to q\bar{q}$. Similarly, two-loop helicity amplitudes for $H \to ggg$ and $H \to q\bar{q}g$ in th elarge $m_t$ limit [71] give access to the gluon-gluon and quark-antiquark splitting functions. The master integrals for these processes involve planar and non-planar two-loop boxes that have been written in terms of two-dimensional harmonic polylogarithms [72, 73] (2dHPL) using differential equations [74]. In the collinear limit, these 2dHPL collapse to harmonic polylogarithms of a single variable [75] that can be re-expressed as Nielsen polylogarithms.

Our paper is organised as follows. In section 2, we define some basic notation for the three- and four-particle processes. Although the full amplitudes factorise, it is often convenient to work at the level of colour ordered amplitudes [69] and [76]–[80]. For the processes at hand, the colour structure is particularly simple and the colour ordered amplitudes are straightforwardly obtained by stripping away the colour matrices. The collinear singularities of the full amplitude are then obtained by restoring these colour matrices. The collinear limit of the colour ordered amplitudes is described in section 3 where we define the splitting functions in terms of the momenta and helicities of the collinear particles and also specify how the collinear limit is approached. The procedure for renormalising the splitting functions is also described. We turn to the quark-gluon collinear limit in section 4 and treat the gluon-gluon and quark-antiquark collinear limits in sections 5 and 6. In each case we show how the general helicity structure of the four-particle amplitude behaves in the singular limit and give results for the tree, one-loop and two-loop splitting functions. In the case of gluon-gluon splitting, we recover the QCD results of ref. [1] in the 't Hooft-Veltman scheme where external particles are treated in 4-dimensions while the particles internal to the loop are in $(4-2\epsilon)$-dimensions. We examine the limits as $w \to 0$ and $w \to 1$ in section 7 and show that the soft behaviour of the two-loop splitting functions is well behaved and consistant with the expected behaviour. As a by product of our calculation, we provide an expression for the two-loop soft splitting function in section 8. Finally, our main results are summarized in Section 9 and accompanied by concluding remarks. The one-loop singularity operators, $\boldsymbol{I}_1$ in Catani's language, for each of the processes studied here are detailed in the appendix.

## 2. Notation

We consider the soft and collinear limits of two-loop matrix elements by taking the limits of known matrix elements. In general, we consider the decay of a massive object $X$ into three partons,

$$X(q) \longrightarrow a(p_a) + b(p_b) + c(p_c)\,, \tag{2.1}$$

where $X$ is a virtual photon or Higgs boson and $a$, $b$ and $c$ are massless partons. Specifically, we have

$$\gamma^* \longrightarrow q + \bar{q} + g\,, \tag{2.2}$$
$$H \longrightarrow g + g + g\,, \tag{2.3}$$
$$H \longrightarrow q + g + \bar{q}\,. \tag{2.4}$$

In the collinear limits, the matrix elements factorise onto the two particle final states,

$$X(q) \longrightarrow A(p_A) + B(p_B)\,, \tag{2.5}$$

or equivalently

$$\gamma^* \longrightarrow Q + \bar{Q}\,, \tag{2.6}$$
$$H \longrightarrow G + G\,. \tag{2.7}$$

The kinematics of these processes is fully described by the invariants

$$s_{ab} = (p_a + p_b)^2\,, \qquad s_{ac} = (p_a + p_c)^2\,, \qquad s_{bc} = (p_b + p_c)^2\,, \tag{2.8}$$

and

$$s_{abc} = (p_a + p_b + p_c)^2 = (p_A + p_B)^2 = s_{AB}\,, \tag{2.9}$$

which fulfil

$$s_{ab} + s_{ac} + s_{bc} \equiv s_{abc} \equiv s_{AB}\,. \tag{2.10}$$

The *unrenormalised* amplitude for process $\mathcal{P}$ is a vector in colour space and can be written in terms of a colour-stripped amplitude,

$$|\mathcal{M}_{\mathcal{P}}\rangle = \boldsymbol{C}_{\mathcal{P}} \mathcal{A}_{\mathcal{P}}\,. \tag{2.11}$$

For the processes at hand the overall colour structures are given by,

$$\boldsymbol{C}_{\gamma^* \to Q\bar{Q}} = \boldsymbol{\delta}_{AB}\,, \tag{2.12}$$
$$\boldsymbol{C}_{\gamma^* \to q\bar{q}g} = \boldsymbol{T}^c_{ab}\,, \tag{2.13}$$
$$\boldsymbol{C}_{H \to GG} = Tr(\boldsymbol{T}^A \boldsymbol{T}^B)\,, \tag{2.14}$$
$$\boldsymbol{C}_{H \to ggg} = \boldsymbol{f}^{abc}\,, \tag{2.15}$$
$$\boldsymbol{C}_{H \to q\bar{q}g} = \boldsymbol{T}^c_{ab}\,, \tag{2.16}$$

where upper and lower indices represent colour indices in the adjoint and fundamental representations respectively. Note that the colour ordering of the two- and three-particle amplitudes is trivial here. The normalisation $Tr(\boldsymbol{T}^A \boldsymbol{T}^B) = \boldsymbol{\delta}^{AB}/2$ is used throughout. While the explicit calculations presented here will be derived from the processes of eqs. (2.2) –(2.7), the universal factorisation properties of QCD amplitudes ensure that the singular limits will be applicable to processes with more particles and more complicated colour structures.

The colour ordered amplitudes have a perturbative expansion in terms of the colour stripped $i$-loop amplitude $|\mathcal{A}_{\mathcal{P}}^{(i)}\rangle$,

$$\mathcal{A}_{\mathcal{P}} = C_{\mathcal{P}} \left[ \mathcal{A}_{\mathcal{P}}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) \mathcal{A}_{\mathcal{P}}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{A}_{\mathcal{P}}^{(2)} + \mathcal{O}(\alpha_s^3) \right], \tag{2.17}$$

where $C_{\mathcal{P}}$ contains the overall couplings such that

$$C_{\gamma^* \to Q\bar{Q}} = \sqrt{4\pi\alpha}\,, \tag{2.18}$$
$$C_{\gamma^* \to q\bar{q}g} = \sqrt{4\pi\alpha}\sqrt{4\pi\alpha_s}\,, \tag{2.19}$$
$$C_{H \to GG} = C_1\,, \tag{2.20}$$
$$C_{H \to ggg} = C_1\sqrt{4\pi\alpha_s}\,, \tag{2.21}$$
$$C_{H \to q\bar{q}g} = C_1\sqrt{4\pi\alpha_s}\,, \tag{2.22}$$

and where $C_1$ is the effective Higgs-gluon-gluon coupling.

## 3. The collinear limit

When two particles become collinear, colour ordered amplitudes factorise in a universal way [6, 23, 25]. If we focus on particles $a$ and $b$ with helicities $\lambda_a$ and $\lambda_b$ becoming collinear so that $s_{ab} \to 0$ and forming particle $A$ with helicity $\lambda_A$, then at tree-level we find that,

$$\mathcal{A}^{(0)}(a^{\lambda_a}, b^{\lambda_b}, \ldots) \xrightarrow{a||b} \sum_{\lambda_A = \pm} \mathrm{Split}^{A \to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(0)}(A^{\lambda_A}, \ldots). \tag{3.1}$$

The factorisation of the $(n+1)$-particle colour ordered tree amplitude into a $n$-particle tree amplitude multiplied by the splitting function is illustrated in figure 1. The singular factors are all contained in $\mathrm{Split}^{A \to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})$ which behaves as $1/\sqrt{s_{ab}}$. The momentum of the slightly off-shell particle $A$ links the two factors,

$$p_A = p_a + p_b\,,$$

while the helicity $\lambda_A$ is summed over. However, for some helicity combinations, such as the case of a positive helicity gluon splitting into two negative helicity gluons, the tree splitting function vanishes.

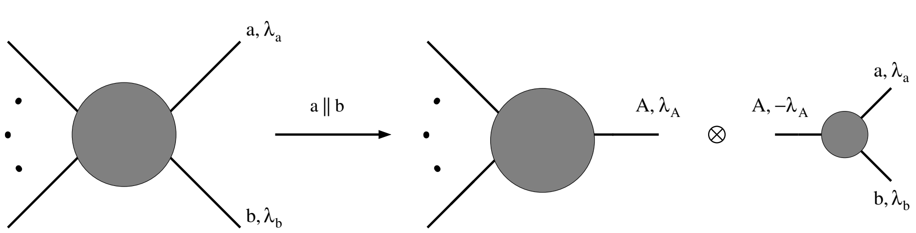


**Figure 1:** The collinear behaviour of a tree-level amplitude. Particles $a$ and $b$ form (a slightly offshell) particle $A$. The singular behaviour is encapsulated in the right hand vertex and is denoted by $\mathrm{Split}^{A \to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})$.

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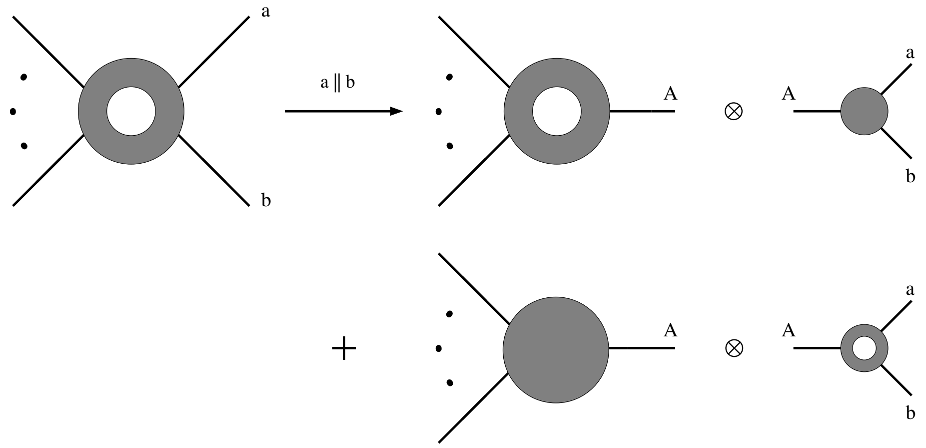


**Figure 2:** The collinear behaviour of a one-loop amplitude.

Similarly, at one- and two-loops, the colour stripped amplitudes factorise as follows,

$$\begin{aligned}
\mathcal{A}^{(1)}(a^{\lambda_a}, b^{\lambda_b}, \ldots) \stackrel{a||b}{\longrightarrow} & \sum_{\lambda_A=\pm} \mathrm{Split}^{A\to ab,(1)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(0)}(A^{\lambda_A}, \ldots) + \\
& + \sum_{\lambda_A=\pm} \mathrm{Split}^{A\to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(1)}(A^{\lambda_A}, \ldots) , \qquad (3.2) \\
\mathcal{A}^{(2)}(a^{\lambda_a}, b^{\lambda_b}, \ldots) \stackrel{a||b}{\longrightarrow} & \sum_{\lambda_A=\pm} \mathrm{Split}^{A\to ab,(2)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(0)}(A^{\lambda_A}, \ldots) + \\
& + \sum_{\lambda_A=\pm} \mathrm{Split}^{A\to ab,(1)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(1)}(A^{\lambda_A}, \ldots) + \\
& + \sum_{\lambda_A=\pm} \mathrm{Split}^{A\to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) \mathcal{A}^{(2)}(A^{\lambda_A}, \ldots) , \qquad (3.3)
\end{aligned}$$

where $\mathrm{Split}^{A\to ab,(i)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})$ denotes the $i$-loop splitting function. These factorisations are illustrated in figures 2 and 3.

In addition to the explicit helicity and particle-type dependence, all of the splitting functions also depend on the momentum fraction $w$ carried by each particle,

$$\begin{aligned}
p_a &\to (1-w)p_A + k_\perp - \frac{k_\perp^2}{1-w}\frac{n}{2p_A\cdot n} , \\
p_b &\to wp_A - k_\perp - \frac{k_\perp^2}{w}\frac{n}{2p_A\cdot n} , \\
s_{ab} &= -\frac{k_\perp^2}{w(1-w)} \to 0, \qquad (3.4)
\end{aligned}$$

where $k_\perp$ is the transverse momentum and $n$ is an arbitrary lightlike momentum such that $2Q\cdot k_\perp = 2n\cdot k_\perp = 0$, and implicitly on the dimensional regularisation parameter $\epsilon$.

The tree and one-loop splitting functions have been known for some time [13, 6, 23, 25], however the form of the two-loop splitting function is only known in the case where a gluon split into gluons [1].

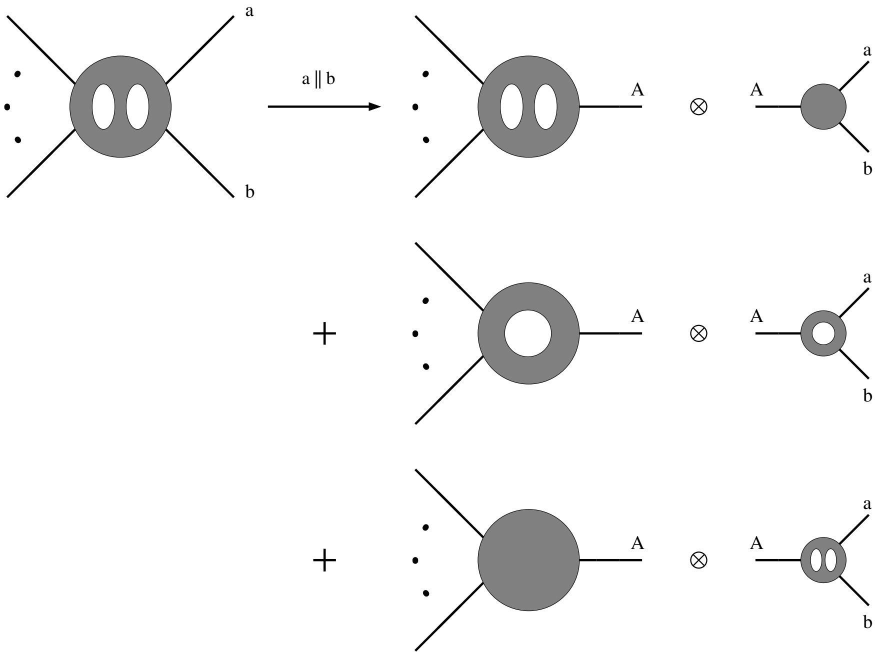


**Figure 3:** The collinear behaviour of a two-loop amplitude.

When the tree splitting function does not vanish, it is customary to factor it out,

$$\mathrm{Split}^{A\to ab,(1)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) = r_S^{A\to ab,(1)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})\, \mathrm{Split}^{A\to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}), \quad (3.5)$$

such that all of the infrared singularities lie in the ratio factor $r_S^{A\to ab,(1)}$. This factor contains all of the non-trivial $w$ dependence as well as the scale violating factor,

$$\left(\frac{4\pi\mu^2}{s_{ab}}\right)^{\epsilon}$$

that modifies the $1/\sqrt{s_{ab}}$ behaviour of the tree-level splitting function. $r_S^{A\to ab,(1)}$ also contains the infrared poles produced by soft and collinear particles circulating in the loop. Explicit formulae, valid to all orders in the dimensional regularisation parameter are given in refs. [23, 25].

Similarly, at two-loops, when the tree-level splitting function does not vanish, the singularities are factored into $r_S^{A\to ab,(2)}$,

$$\mathrm{Split}^{A\to ab,(2)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b}) = r_S^{A\to ab,(2)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})\, \mathrm{Split}^{A\to ab,(0)}(-\lambda_A, a^{\lambda_a}, b^{\lambda_b})\,. \quad (3.6)$$

Catani [63] has shown how to organize the infrared pole structure of the two-loop contributions renormalized in the $\overline{\mathrm{MS}}$ scheme in terms of the tree and renormalized one-loop amplitudes. Motivated by the structure of Catani's formula and by the results for planar amplitudes in maximally supersymmetric Yang-Mills theory [81], we write the singular behaviour of the *unrenormalised* $r_S^{A\to ab,(2)}$ as (dropping the helicity labels),

$$\begin{aligned} r_S^{A\to ab,(2)}(\epsilon) = {}& \frac{1}{2}\left(r_S^{A\to ab,(1)}(\epsilon)\right)^2 + \frac{e^{-\epsilon\gamma}c_\Gamma(\epsilon)}{c_\Gamma(2\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right) r_S^{A\to ab,(1)}(2\epsilon) + \Delta H^{A\to ab,(2)}(\epsilon) + \\ & + r_S^{A\to ab,(2),\mathrm{fin}}(\epsilon) + \mathcal{O}(\epsilon)\,, \end{aligned} \quad (3.7)$$

where the Euler constant$\gamma = 0.5772\ldots$ and

$$c_\Gamma(\epsilon) = \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{3.8}$$

The first two terms contain all infrared singularities from $\mathcal{O}(1/\epsilon^4)$ through to $\mathcal{O}(1/\epsilon^2)$ while the remaining single poles are absorbed in $\Delta H^{A\to ab,(2)}(\epsilon)$,

$$\Delta H^{A\to ab,(2)}(\epsilon) = \frac{e^{-\epsilon\gamma}c_\Gamma(\epsilon)}{4\epsilon}\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^{2\epsilon}(w(1-w))^{-2\epsilon}\left(H_a^{(2)}+H_b^{(2)}-H_A^{(2)}-\beta_0 K+\beta_1\right). \tag{3.9}$$

For splitting functions involving only two collinear particles, there are no colour correlations in $\Delta H^{A\to ab,(2)}(\epsilon)$. Here, $H_q$ ($H_{\bar{q}}$) and $H_g$ are the usual constants appearing in Catani's *renormalised* formula,

$$\begin{aligned} H_q^{(2)} = H_{\bar{q}}^{(2)} &= \left(\frac{\pi^2}{2} - 6\,\zeta_3 - \frac{3}{8}\right)C_F^2 + \left(\frac{13}{2}\zeta_3 + \frac{245}{216} - \frac{23}{48}\pi^2\right)C_A C_F + \\ &\quad + \left(-\frac{25}{54} + \frac{\pi^2}{12}\right)T_R N_F C_F\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} H_g^{(2)} &= \frac{20}{27}T_R^2 N_F^2 + T_R C_F N_F - \left(\frac{\pi^2}{36} + \frac{58}{27}\right)T_R N_F C_A + \\ &\quad + \left(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11}{144}\pi^2\right)C_A^2\,, \end{aligned} \tag{3.11}$$

while the constants $\beta_0$, $\beta_1$ and $K$ are

$$\beta_0 = \frac{11C_A - 2N_F}{6}\,, \tag{3.12}$$

$$\beta_1 = \frac{17C_A^2 - 10C_A T_R N_F - 6C_F T_R N_F}{6}\,, \tag{3.13}$$

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_R N_F\,. \tag{3.14}$$

For SU($N$) gauge theory, $C_A = N$, $C_F = (N^2-1)/2/N$ and $T_R = 1/2$. Note that while the pole contribution to $\Delta H^{A\to ab,(2)} = \left(H_a^{(2)}+H_b^{(2)}-H_A^{(2)}-\beta_0 K+\beta_1\right)/\epsilon$ is fixed, there is considerable freedom in defining the $\mathcal{O}(\epsilon)$ contribution to the factor multiplying the pole. Different choices affect the finite remainder. We choose this particular factor since it regulates the $s_{ab}\to 0$, $w\to 0$ and $w\to 1$ limits of the splitting function.

The remaining finite contributions are contained in $r_S^{A\to ab,(2),\mathrm{fin}}(\epsilon)$. As we will demonstrate, explicit calculation shows that eq. (3.7) fixes the infrared pole structure of the splitting function exactly, leaving only $r_S^{A\to ab,(2),\mathrm{fin}}(\epsilon)$ to be determined.

### 3.1 Ultraviolet renormalization

The splitting functions have the perturbative expansion,

$$\mathrm{Split}^{A\to ab} = \sqrt{4\pi\alpha_s}\left[\mathrm{Split}^{A\to ab,(0)} + \left(\frac{\alpha_s}{2\pi}\right)\mathrm{Split}^{A\to ab,(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2\mathrm{Split}^{A\to ab,(2)} + \mathcal{O}(\alpha_s^3)\right]. \tag{3.15}$$

The renormalization of the splitting function is carried out by replacing the bare coupling $\alpha_s$ with the renormalized coupling $\alpha_s(\mu^2)$, evaluated at the renormalization scale $\mu^2$

$$\alpha_s \mu_0^{2\epsilon} S_\epsilon = \alpha_s(\mu^2)\mu^{2\epsilon}\left[1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s(\mu^2)}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right], \qquad (3.16)$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma},$$

and $\mu_0^2$ is the mass parameter introduced in dimensional regularization [82, 83, 70] to maintain a dimensionless coupling in the bare QCD lagrangian density.

Applying the renormalisation procedure, we have

$$\begin{aligned}
\mathrm{Split}^{A\to ab,(0),R} &= \mathrm{Split}^{A\to ab,(0)}, \\
\mathrm{Split}^{A\to ab,(1),R} &= S_\epsilon^{-1}\,\mathrm{Split}^{A\to ab,(1)} - \frac{\beta_0}{2\epsilon}\,\mathrm{Split}^{A\to ab,(0)}, \qquad (3.17)\\
\mathrm{Split}^{A\to ab,(2),R} &= S_\epsilon^{-2}\,\mathrm{Split}^{A\to ab,(2)} - \frac{3\beta_0}{2\epsilon}S_\epsilon^{-1}\,\mathrm{Split}^{A\to ab,(1)} - \left(\frac{\beta_1}{4\epsilon} - \frac{3\beta_0^2}{8\epsilon^2}\right)\mathrm{Split}^{A\to ab,(0)}.
\end{aligned}$$

Equivalently, for the amplitudes where the tree splitting function does not vanish,

$$\begin{aligned}
r_S^{A\to ab,(1),R} &= S_\epsilon^{-1} r_S^{A\to ab,(1)} - \frac{\beta_0}{2\epsilon}, \\
r_S^{A\to ab,(2),R} &= S_\epsilon^{-2} r_S^{A\to ab,(2)} - \frac{3\beta_0}{2\epsilon}S_\epsilon^{-1} r_S^{A\to ab,(1)} - \left(\frac{\beta_1}{4\epsilon} - \frac{3\beta_0^2}{8\epsilon^2}\right). \qquad (3.18)
\end{aligned}$$

Applying this transformation to the unrenormalised splitting ratios, we find that,

$$\begin{aligned}
r_S^{A\to ab,(2),R}(\epsilon) = {}& \frac{1}{2}\left(r_S^{A\to ab,(1),R}(\epsilon)\right)^2 - \frac{\beta_0}{\epsilon} r_S^{A\to ab,(1),R}(\epsilon) + \\
&+ \frac{e^{-\epsilon\gamma} c_\Gamma(\epsilon)}{c_\Gamma(2\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right) r_S^{A\to ab,(1),R}(2\epsilon) + \\
&+ \Delta H^{A\to ab,(2),R}(\epsilon) + r_S^{A\to ab,(2),R,\mathrm{fin}}(\epsilon) + \mathcal{O}(\epsilon), \qquad (3.19)
\end{aligned}$$

where,

$$\Delta H^{A\to ab,(2),R}(\epsilon) = \frac{e^{-\epsilon\gamma} c_\Gamma(\epsilon)}{4\epsilon}\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^{2\epsilon}(w(1-w))^{-2\epsilon}\left(H_a^{(2)} + H_b^{(2)} - H_A^{(2)}\right) \qquad (3.20)$$

together with a modification of the finite remainder, $r_S^{A\to ab,(2),R,\mathrm{fin}}(\epsilon)$. Up to trivial gamma function coefficients, which only modify the finite remainder, eq. (3.19) has the same structure as that given for the renormalised gluon splitting function in ref. [1] and also Catani's formula for the general structure of renormalised two-loop amplitudes [63]. Throughout the remainder of the paper we will work with unrenormalised amplitudes.

## 4. The quark-gluon splitting amplitudes

The unrenormalized amplitude $|\mathcal{M}\rangle$ for the virtual photon initiated processes, (2.6) and (2.2), can be written as

$$
\begin{aligned}
|\mathcal{M}_{\gamma^*\to Q\bar{Q}}\rangle &= \boldsymbol{\delta}_{AB}\, V^\mu\, \mathrm{S}_\mu(Q;\bar{Q})\,, \\
|\mathcal{M}_{\gamma^*\to q\bar{q}g}\rangle &= \boldsymbol{T}^c_{ab}\, V^\mu\, \mathrm{S}_\mu(q;g;\bar{q})\,,
\end{aligned} \tag{4.1}
$$

where $V^\mu$ represents the lepton current and $\mathrm{S}_\mu$ denotes the colour ordered hadron current. Using the Weyl-van der Waerden spinor notation (see ref. [69]), the hadronic current $\mathrm{S}_\mu$ is related to the fixed helicity currents, $\mathrm{S}_{\dot{A}B}$, by

$$\mathrm{S}_\mu(Q+;\overline{Q}-) = R^V_{f_1 f_2}\, \sigma_\mu^{\dot{A}B}\, \mathrm{S}_{\dot{A}B}(Q+;\overline{Q}-)\,, \tag{4.2}$$

$$\mathrm{S}_\mu(Q-;\overline{Q}+) = L^V_{f_1 f_2}\, \sigma_\mu^{\dot{A}B}\, \mathrm{S}_{\dot{A}B}(Q-;\overline{Q}+)\,, \tag{4.3}$$

$$\mathrm{S}_\mu(q+;g\lambda;\overline{q}-) = R^V_{f_1 f_2}\sqrt{2}\, \sigma_\mu^{\dot{A}B}\, \mathrm{S}_{\dot{A}B}(q+;g\lambda;\overline{q}-)\,, \tag{4.4}$$

$$\mathrm{S}_\mu(q-;g\lambda;\overline{q}+) = L^V_{f_1 f_2}\sqrt{2}\, \sigma_\mu^{\dot{A}B}\, \mathrm{S}_{\dot{A}B}(q-;g\lambda;\overline{q}+)\,. \tag{4.5}$$

Labelling the particle momenta by their type, and dropping terms that vanish when contracted with the lepton current, we find that

$$\mathrm{S}_{\dot{A}B}(Q+;\overline{Q}-) = A\ Q_{\dot{A}}\overline{Q}_B \tag{4.6}$$

$$\mathrm{S}_{\dot{A}B}(q+;g+;\overline{q}-) = \alpha(y,z)\ \frac{q_{\dot{A}D}\bar{q}^D\bar{q}_B}{\langle qg\rangle\langle g\bar{q}\rangle} + \beta(y,z)\ \frac{g_{\dot{A}D}\bar{q}^D\bar{q}_B}{\langle qg\rangle\langle g\bar{q}\rangle} + \gamma(y,z)\ \frac{q_{\dot{C}B}g^{\dot{C}}g_{\dot{A}}}{\langle qg\rangle\langle g\bar{q}\rangle^*}\,, \tag{4.7}$$

$$\mathrm{S}_{\dot{A}B}(q+;g-;\overline{q}-) = -\alpha(z,y)\ \frac{\bar{q}_{\dot{D}B}q^{\dot{D}}q_{\dot{A}}}{\langle qg\rangle^*\langle g\bar{q}\rangle^*} - \beta(z,y)\ \frac{g_{\dot{D}B}q^{\dot{D}}q_{\dot{A}}}{\langle qg\rangle^*\langle g\bar{q}\rangle^*} - \gamma(z,y)\ \frac{\bar{q}_{\dot{A}C}g^{C}g_B}{\langle \bar{q}g\rangle^*\langle gq\rangle}\,. \tag{4.8}$$

In these expressions, $\langle ij\rangle = \langle i-|j+\rangle$ and $[ij] = \langle i+|j-\rangle$ where $|i\lambda\rangle$ is the Weyl spinor for a massless particle with momentum $i$ and helicity $\lambda$. The spinor products are antisymmetric and satisfy $[ij] = -\langle ij\rangle^*$ and $\langle ij\rangle[ji] = s_{ij}$.

The currents with the quark helicities flipped follow from parity conservation:

$$
\begin{aligned}
\mathrm{S}_{\dot{A}B}(Q-;\overline{Q}+) &= (\,\mathrm{S}_{\dot{B}A}(Q+;\overline{Q}-))^*\,, \\
\mathrm{S}_{\dot{A}B}(q-;g\lambda;\overline{q}+) &= (\,\mathrm{S}_{\dot{B}A}(q+;g(-\lambda);\overline{q}-))^*\,.
\end{aligned} \tag{4.9}
$$

The coefficients $A$ and $\Omega$ ($\Omega = \alpha,\beta,\gamma$) contain all of the integrals over loop momenta and ultimately determine the infrared structure of the amplitude. $A$ depends only on the invariant mass of the virtual photon $s_{AB}$, while $\Omega$ depends on $y = s_{qg}/s_{qg\bar{q}}$ and $z = s_{g\bar{q}}/s_{qg\bar{q}}$. Each coefficient is a linear combination of master loop integrals [72, 73] with coefficients that are rational functions of the scale invariants and the spacetime dimension. Expansions of the renormalised $\Omega$ around $\epsilon = 0$ are given in ref. [51]–[53].

Taking the quark-gluon collinear limit corresponds to $y = s_{qg}/s_{qg\bar{q}} \to 0$, with the quark and gluon carrying momentum fractions $(1-w)Q$ and $wQ$ respectively. When the quark and gluon both have positive helicity we find that,

$$\mathrm{S}_{\dot{A}B}(q+;g+;\overline{q}-) \xrightarrow{q||g} \frac{1}{\sqrt{w}\ \langle qg\rangle}\left[\alpha(y,z)\Big|_{y\to 0}(1-w) + \beta(y,z)\Big|_{y\to 0} w\right] Q_{\dot{A}}\overline{Q}_B\,. \tag{4.10}$$

Similarly, when the quark has positive helicity but the gluon has negative helicity, we find,

$$\mathrm{S}_{\dot{A}B}(q+;g-;\overline{q}-) \xrightarrow{q||g} -\frac{1-w}{\sqrt{w}\,[qg]}\left[\alpha(z,y)\Big|_{y\to 0}\right] Q_{\dot{A}}\overline{Q}_B\,. \tag{4.11}$$

We see that in both cases we recover the two-particle helicity structure of eq. (4.10).

In the antiquark-gluon collinear limit, $z = s_{g\bar{q}}/s_{qg\bar{q}} \to 0$ and the antiquark and gluon carrying momentum fractions $(1-w)Q$ and $wQ$ respectively. When the antiquark has negative helicity then,

$$\mathrm{S}_{\dot{A}B}(q+;g-;\overline{q}-) \xrightarrow{\bar{q}||g} -\frac{1}{\sqrt{w}\,[g\bar{q}]}\left[\alpha(z,y)\Big|_{z\to 0}(1-w)+\beta(z,y)\Big|_{z\to 0} w\right] Q_{\dot{A}}\overline{Q}_B\,, \tag{4.12}$$

$$\mathrm{S}_{\dot{A}B}(q+;g+;\overline{q}-) \xrightarrow{\bar{q}||g} \frac{1-w}{\sqrt{w}\,\langle g\bar{q}\rangle}\left[\alpha(y,z)\Big|_{z\to 0}\right] Q_{\dot{A}}\overline{Q}_B\,. \tag{4.13}$$

The helicity amplitude coefficients have the perturbative expansions,

$$A = C_{\gamma^*\to Q\bar{Q}}\left[A^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)A^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 A^{(2)} + \mathcal{O}(\alpha_s^3)\right], \tag{4.14}$$

and

$$\Omega(y,z) = C_{\gamma^*\to q\bar{q}g}\left[\Omega^{(0)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)\Omega^{(1)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)^2\Omega^{(2)}(y,z) + \mathcal{O}(\alpha_s^3)\right], \tag{4.15}$$

for $\Omega = \alpha,\beta,\gamma$.

### 4.1 The tree-level quark-gluon splitting amplitudes

At leading order,

$$\alpha^{(0)}(y,z) = \beta^{(0)}(y,z) = 1 \qquad \text{and} \qquad A^{(0)} = 1\,. \tag{4.16}$$

Bearing in mind that the splitting function defined in eq. (3.1) relates the colour ordered amplitudes, comparing eqs. (4.10) and (4.11) with eq. (4.6), we can immediately read off the tree-level quark-gluon splitting functions for colour stripped amplitudes,

$$\begin{aligned}
\mathrm{Split}^{Q\to qg,(0)}(-,q^+,g^+) &= \frac{\sqrt{2}}{\sqrt{w}\,\langle qg\rangle}\,,\\
\mathrm{Split}^{Q\to qg,(0)}(-,q^+,g^-) &= \frac{-\sqrt{2}\,(1-w)}{\sqrt{w}\,[qg]}\,.
\end{aligned} \tag{4.17}$$

The factors of $\sqrt{2}$ are remnants of the definition of the helicity current in eq. (4.4). In addition, the soft gluon singularity as $w\to 0$ is explicit.

Splitting functions where $Q$ has negative helicity are obtained by complex conjugation according to eq. (4.9),

$$\begin{aligned}
\mathrm{Split}^{Q\to qg,(0)}(+,q^-,g^-) &= \frac{-\sqrt{2}}{\sqrt{w}\,[qg]}\,,\\
\mathrm{Split}^{Q\to qg,(0)}(+,q^-,g^+) &= \frac{\sqrt{2}\,(1-w)}{\sqrt{w}\,\langle qg\rangle}\,.
\end{aligned} \tag{4.18}$$

Similarly, the tree splitting functions for an antiquark $\bar{Q}$ to split into a gluon $g$ and antiquark $\bar{q}$ are given by,

$$\mathrm{Split}^{\bar{Q}\to g\bar{q},(0)}(+,g^+,\bar{q}^-) = \frac{\sqrt{2}}{\sqrt{w}\,[g\bar{q}]}, \qquad \mathrm{Split}^{\bar{Q}\to g\bar{q},(0)}(+,g^-,\bar{q}^-) = -\frac{\sqrt{2}(1-w)}{\sqrt{w}\,\langle g\bar{q}\rangle} \tag{4.19}$$

$$\mathrm{Split}^{\bar{Q}\to g\bar{q},(0)}(-,g^-,\bar{q}^+) = -\frac{\sqrt{2}}{\sqrt{w}\,\langle g\bar{q}\rangle}, \qquad \mathrm{Split}^{\bar{Q}\to g\bar{q},(0)}(-,g^+,\bar{q}^+) = \frac{\sqrt{2}(1-w)}{\sqrt{w}\,[g\bar{q}]}. \tag{4.20}$$

### 4.2 The one-loop quark-gluon splitting amplitudes

The unrenormalised one-loop splitting functions defined in eq. (3.5) are presented in ref. [25] in a variety of schemes to all orders in $\epsilon$, and we reproduce them here in the 't Hooft-Veltman regularisation scheme ($\delta_R = 1$) for the sake of completeness,

$$\begin{aligned} r_S^{Q\to qg,(1)}(\mp,q^\pm,g^\pm) &= r_S^{\overline{Q}\to g\bar{q},(1)}(\mp,g^\pm,\bar{q}^\pm) = \\ &= -c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{qg}}\right)^\epsilon \left\{\frac{N}{2\epsilon^2}\left[1-\sum_{m=1}^{\infty}\epsilon^m\left[\mathrm{Li}_m\left(\frac{1-w}{-w}\right)-\frac{1}{N^2}\mathrm{Li}_m\left(\frac{-w}{1-w}\right)\right]\right]-\right.\\ &\qquad \left. -\left(\frac{N^2+1}{N}\right)\frac{w}{4(1-2\epsilon)}\right\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} r_S^{Q\to qg,(1)}(\mp,q^\pm,g^\mp) &= r_S^{\overline{Q}\to g\bar{q},(1)}(\mp,g^\pm,\bar{q}^\mp) = \\ &= -c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{qg}}\right)^\epsilon \left\{\frac{N}{2\epsilon^2}\left[1-\sum_{m=1}^{\infty}\epsilon^m\left[\mathrm{Li}_m\left(\frac{1-w}{-w}\right)-\frac{1}{N^2}\mathrm{Li}_m\left(\frac{-w}{1-w}\right)\right]\right]\right\}. \end{aligned} \tag{4.22}$$

As usual, the polylogarithms $\mathrm{Li}_n(w)$ are defined by

$$\begin{aligned} \mathrm{Li}_n(w) &= \int_0^w \frac{dt}{t}\mathrm{Li}_{n-1}(t) \qquad \text{for } n=3,4,\ldots \\ \mathrm{Li}_2(w) &= -\int_0^w \frac{dt}{t}\ln(1-t)\,. \end{aligned} \tag{4.23}$$

We have checked that our procedure of taking the collinear limit of the explicit one-loop amplitudes written in terms of 2dHPL reproduces these expressions through to $\mathcal{O}(\epsilon^2)$. Note that, unlike ref. [25], in our notation the gluon always carries a momentum fraction $w$. Note also that because of our different normalisations, the one-loop splitting function differs by a factor of $N/2$ compared to ref. [25].

### 4.3 The two-loop quark-gluon splitting amplitudes

Explicit calculations for the unrenormalised two-loop coefficients, that independently satisfy the pole structure predicted by Catani (see appendix A) can be found in ref. [52]. Using FORM [84] and MAPLE to take the limits of the 2dHPL, we find that the unrenormalised splitting functions $r_S^{Q\to qg,(2)}(\epsilon)$ have a pole structure determined by eq. (3.7). The finite remainders are given by,

$$r_S^{Q\to qg,(2),\mathrm{fin}}(\mp,q^\pm,g^\pm) = r_S^{\overline{Q}\to g\bar{q},(2),\mathrm{fin}}(\mp,g^\pm,\bar{q}^\pm) =$$

$$
\begin{aligned}
&= +N^2\left(-\frac{3}{2}\mathrm{Li}_4(w) - \frac{5}{4}\mathrm{Li}_4(1-w) + \frac{3}{2}\mathrm{Li}_4\left(\frac{w}{w-1}\right) + \mathrm{Li}_3(w)\ln(w) + \frac{1}{3}\mathrm{Li}_3(w) + \right.\\
&\qquad + \mathrm{Li}_3(1-w)\ln(w) + \frac{13}{24}\mathrm{Li}_3(1-w) + \frac{1}{8}\mathrm{Li}_2(w)^2 + \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) -\\
&\qquad - \frac{1}{6}\mathrm{Li}_2(w)\ln(w) + \frac{3}{4}\mathrm{Li}_2(w)\zeta_2 - \frac{1}{8}\mathrm{Li}_2(w) + \frac{1}{2}\ln(w)^2\ln(1-w)^2 -\\
&\qquad - \frac{1}{4}\ln(w)\ln(1-w)^3 - \frac{3}{4}\ln(w)\ln(1-w)\zeta_2 - \frac{1}{8}\ln(w)\ln(1-w) - \ln(w)\zeta_3 +\\
&\qquad + \frac{1}{16}\ln(1-w)^4 + \frac{3}{4}\ln(1-w)^2\zeta_2 + \frac{1}{4}\ln(1-w)\zeta_3 + \frac{55}{48}\ln(1-w)\zeta_2 -\\
&\qquad \left. - \frac{193}{108}\ln(1-w) + \frac{29}{32}\zeta_4 + \frac{527}{144}\zeta_3 + \frac{335}{288}\zeta_2 - \frac{571}{324}\right) +\\
&\quad + \left(-2\mathrm{Li}_4(w) - \frac{7}{4}\mathrm{Li}_4(1-w) + \mathrm{Li}_4\left(\frac{w}{w-1}\right) + \mathrm{Li}_3(w)\ln(w) + \frac{1}{4}\mathrm{Li}_3(w)\ln(1-w) + \right.\\
&\qquad + \frac{3}{2}\mathrm{Li}_3(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) + \frac{9}{4}\mathrm{Li}_3(1-w) -\\
&\qquad - \frac{1}{8}\mathrm{Li}_2(w)^2 - \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) - \frac{3}{4}\mathrm{Li}_2(w)\ln(w) + \frac{3}{4}\mathrm{Li}_2(w)\zeta_2 -\\
&\qquad - \frac{3}{4}\mathrm{Li}_2(1-w)\ln(1-w) + \frac{1}{4}\ln(w)^2\ln(1-w)^2 - \frac{1}{6}\ln(w)\ln(1-w)^3 -\\
&\qquad - \frac{1}{4}\ln(w)\ln(1-w)\zeta_2 - \frac{1}{2}\ln(w)\zeta_3 + \frac{1}{24}\ln(1-w)^4 + \frac{1}{2}\ln(1-w)^2\zeta_2 +\\
&\qquad + \frac{3}{4}\ln(1-w)\zeta_3 + \frac{11}{48}\ln(1-w)\zeta_2 - \frac{323}{216}\ln(1-w) + \frac{7}{4}\zeta_4 - \frac{9}{4}\zeta_3 +\\
&\qquad + w\left(-\frac{3}{4}\mathrm{Li}_3(w) - \frac{3}{4}\mathrm{Li}_3(1-w) + \frac{1}{4}\mathrm{Li}_2(w)\ln(w) + \frac{1}{4}\mathrm{Li}_2(w) + \right.\\
&\qquad\quad + \frac{1}{4}\mathrm{Li}_2(1-w)\ln(1-w) + \frac{1}{8}\ln(w)\ln(1-w) + \frac{1}{4}\ln(w)\zeta_2 + \frac{3}{8}\ln(w) +\\
&\qquad\quad \left. + \frac{1}{4}\ln(1-w)\zeta_2 - \frac{3}{8}\ln(1-w) - \frac{1}{8}\zeta_2 + \frac{3}{32}\right) +\\
&\qquad + w^2\left(-\frac{3}{8}\mathrm{Li}_3(w) - \frac{3}{8}\mathrm{Li}_3(1-w) + \frac{1}{8}\mathrm{Li}_2(w)\ln(w) + \frac{1}{8}\mathrm{Li}_2(1-w)\ln(1-w) + \right.\\
&\qquad \left.\left. + \frac{1}{8}\ln(w)\zeta_2 + \frac{1}{8}\ln(1-w)\zeta_2 - \frac{1}{32}\right)\right) +\\
&\quad + \frac{1}{N^2}\left(-\frac{1}{2}\mathrm{Li}_4(w) - \frac{1}{2}\mathrm{Li}_4(1-w) - \frac{1}{2}\mathrm{Li}_4\left(\frac{w}{w-1}\right) + \frac{1}{4}\mathrm{Li}_3(w)\ln(1-w) - \right.\\
&\qquad - \frac{1}{2}\mathrm{Li}_3(1-w)\ln(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) + \frac{3}{8}\mathrm{Li}_3(1-w) - \frac{1}{4}\mathrm{Li}_2(w)^2 -\\
&\qquad - \frac{1}{2}\mathrm{Li}_2(w)\ln(w)\ln(1-w) + \frac{1}{8}\mathrm{Li}_2(w) - \frac{1}{4}\ln(w)^2\ln(1-w)^2 +\\
&\qquad + \frac{1}{12}\ln(w)\ln(1-w)^3 + \frac{1}{2}\ln(w)\ln(1-w)\zeta_2 + \frac{1}{8}\ln(w)\ln(1-w) + \frac{1}{2}\ln(w)\zeta_3 -\\
&\qquad - \frac{1}{48}\ln(1-w)^4 - \frac{1}{4}\ln(1-w)^2\zeta_2 + \frac{1}{2}\ln(1-w)\zeta_3 - \frac{3}{4}\ln(1-w)\zeta_2 +
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{3}{8}\ln(1-w) + \frac{1}{2}\zeta_4 - \frac{3}{8}\zeta_3 + \\
&\quad + w\biggl(-\frac{3}{4}\mathrm{Li}_3(w) - \frac{3}{4}\mathrm{Li}_3(1-w) + \frac{1}{4}\mathrm{Li}_2(w)\ln(w) + \frac{1}{4}\mathrm{Li}_2(w) + \\
&\qquad + \frac{1}{4}\mathrm{Li}_2(1-w)\ln(1-w) + \frac{1}{8}\ln(w)\ln(1-w) + \frac{1}{4}\ln(w)\zeta_2 + \frac{3}{8}\ln(w) + \\
&\qquad + \frac{1}{4}\ln(1-w)\zeta_2 - \frac{3}{8}\ln(1-w) - \frac{1}{8}\zeta_2 + \frac{3}{32}\biggr) + \\
&\quad + w^2\biggl(-\frac{3}{8}\mathrm{Li}_3(w) - \frac{3}{8}\mathrm{Li}_3(1-w) + \frac{1}{8}\mathrm{Li}_2(w)\ln(w) + \frac{1}{8}\mathrm{Li}_2(1-w)\ln(1-w) + \\
&\qquad + \frac{1}{8}\ln(w)\zeta_2 + \frac{1}{8}\ln(1-w)\zeta_2 - \frac{1}{32}\biggr)\biggr) + \\
&+ NN_F\biggl(-\frac{1}{3}\mathrm{Li}_3(w) - \frac{1}{6}\mathrm{Li}_3(1-w) + \frac{1}{6}\mathrm{Li}_2(w)\ln(w) - \frac{5}{24}\ln(1-w)\zeta_2 + \\
&\qquad + \frac{19}{108}\ln(1-w) - \frac{43}{72}\zeta_3 - \frac{25}{144}\zeta_2 + \frac{65}{648}\biggr) + \\
&+ \frac{N_F}{N}\left(-\frac{1}{24}\ln(1-w)\zeta_2 + \frac{7}{27}\ln(1-w)\right) + \qquad\qquad (4.24)\\
&+ w\left(\frac{\zeta_2}{4}N^2 + \frac{1}{24}\frac{N_F}{N} + \frac{5}{24}N(N-N_F) + \left(\frac{\zeta_2}{4} - \frac{1}{24}\right)\right) + \frac{w(1-w)}{32}(N^2+1)\,,
\end{aligned}
$$

$$
\begin{aligned}
&r_S^{Q\to qg,(2),\mathrm{fin}}(\mp, q^\pm, g^\mp) = r_S^{\overline{Q}\to g\bar{q},(2),\mathrm{fin}}(\mp, g^\pm, \bar{q}^\mp) = \\
&= +N^2\biggl(-\frac{3}{2}\mathrm{Li}_4(w) - \frac{5}{4}\mathrm{Li}_4(1-w) + \frac{3}{2}\mathrm{Li}_4\left(\frac{w}{w-1}\right) + \mathrm{Li}_3(w)\ln(w) + \frac{1}{3}\mathrm{Li}_3(w) + \\
&\qquad + \mathrm{Li}_3(1-w)\ln(w) + \frac{13}{24}\mathrm{Li}_3(1-w) + \frac{1}{8}\mathrm{Li}_2(w)^2 + \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) - \\
&\qquad - \frac{1}{6}\mathrm{Li}_2(w)\ln(w) + \frac{3}{4}\mathrm{Li}_2(w)\zeta_2 - \frac{1}{8}\mathrm{Li}_2(w) + \frac{1}{2}\ln(w)^2\ln(1-w)^2 - \\
&\qquad - \frac{1}{4}\ln(w)\ln(1-w)^3 - \frac{3}{4}\ln(w)\ln(1-w)\zeta_2 - \frac{1}{8}\ln(w)\ln(1-w) - \ln(w)\zeta_3 + \\
&\qquad + \frac{1}{12}\ln(w) + \frac{1}{16}\ln(1-w)^4 + \frac{3}{4}\ln(1-w)^2\zeta_2 + \frac{1}{4}\ln(1-w)\zeta_3 + \\
&\qquad + \frac{55}{48}\ln(1-w)\zeta_2 - \frac{193}{108}\ln(1-w) + \frac{29}{32}\zeta_4 + \frac{527}{144}\zeta_3 + \frac{335}{288}\zeta_2 - \frac{571}{324}\biggr) + \\
&\quad + \biggl(-2\mathrm{Li}_4(w) - \frac{7}{4}\mathrm{Li}_4(1-w) + \mathrm{Li}_4\left(\frac{w}{w-1}\right) + \mathrm{Li}_3(w)\ln(w) + \\
&\qquad + \frac{1}{4}\mathrm{Li}_3(w)\ln(1-w) + \frac{3}{2}\mathrm{Li}_3(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) + \\
&\qquad + \frac{3}{4}\mathrm{Li}_3(1-w) - \frac{1}{8}\mathrm{Li}_2(w)^2 - \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) - \frac{3}{4}\mathrm{Li}_2(w)\ln(w) + \\
&\qquad + \frac{3}{4}\mathrm{Li}_2(w)\zeta_2 + \frac{1}{4}\ln(w)^2\ln(1-w)^2 - \frac{1}{6}\ln(w)\ln(1-w)^3 - \frac{1}{4}\ln(w)\ln(1-w)\zeta_2 - \\
&\qquad - \frac{1}{2}\ln(w)\zeta_3 - \frac{3}{8}\ln(w) + \frac{1}{24}\ln(1-w)^4 + \frac{1}{2}\ln(1-w)^2\zeta_2 + \frac{3}{4}\ln(1-w)\zeta_3 -
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{25}{48}\ln(1-w)\zeta_2 - \frac{323}{216}\ln(1-w) + \frac{7}{4}\zeta_4 - \frac{3}{4}\zeta_3\Bigg) + \\
&+ \frac{1}{N^2}\Bigg( -\frac{1}{2}\mathrm{Li}_4(w) - \frac{1}{2}\mathrm{Li}_4(1-w) - \frac{1}{2}\mathrm{Li}_4\left(\frac{w}{w-1}\right) + \frac{1}{4}\mathrm{Li}_3(w)\ln(1-w) - \\
&\quad - \frac{1}{2}\mathrm{Li}_3(1-w)\ln(w) + \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) + \frac{3}{8}\mathrm{Li}_3(1-w) - \frac{1}{4}\mathrm{Li}_2(w)^2 - \\
&\quad - \frac{1}{2}\mathrm{Li}_2(w)\ln(w)\ln(1-w) + \frac{1}{8}\mathrm{Li}_2(w) - \frac{1}{4}\ln(w)^2\ln(1-w)^2 + \\
&\quad + \frac{1}{12}\ln(w)\ln(1-w)^3 + \frac{1}{2}\ln(w)\ln(1-w)\zeta_2 + \frac{1}{8}\ln(w)\ln(1-w) + \frac{1}{2}\ln(w)\zeta_3 - \\
&\quad - \frac{3}{8}\ln(w) - \frac{1}{48}\ln(1-w)^4 - \frac{1}{4}\ln(1-w)^2\zeta_2 + \frac{1}{2}\ln(1-w)\zeta_3 - \frac{3}{4}\ln(1-w)\zeta_2 + \\
&\quad + \frac{3}{8}\ln(1-w) + \frac{1}{2}\zeta_4 - \frac{3}{8}\zeta_3\Bigg) + \\
&+ NN_F\Bigg( -\frac{1}{3}\mathrm{Li}_3(w) - \frac{1}{6}\mathrm{Li}_3(1-w) + \frac{1}{6}\mathrm{Li}_2(w)\ln(w) - \frac{1}{12}\ln(w) - \frac{5}{24}\ln(1-w)\zeta_2 + \\
&\quad + \frac{19}{108}\ln(1-w) - \frac{43}{72}\zeta_3 - \frac{25}{144}\zeta_2 + \frac{65}{648}\Bigg) + \\
&+ \frac{N_F}{N}\left( -\frac{1}{24}\ln(1-w)\zeta_2 + \frac{7}{27}\ln(1-w)\right) - \\
&- N^2\Bigg( \frac{w(4-3w)}{4(1-w)^2}\left(\mathrm{Li}_3(w) - \zeta_3 - \frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w) - \zeta_2\right)\right) + \\
&\quad + \frac{w}{8(1-w)}\left(-\mathrm{Li}_2(1-w) + 2\zeta_2\right)\Bigg) + \\
&+ \frac{1}{N^2}\Bigg( \frac{w(3w-2)}{4(1-w)^2}\left(\mathrm{Li}_3(w) - \zeta_3 - \frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w) - \zeta_2\right)\right) + \\
&\quad + \frac{w}{8(1-w)}\left(-\mathrm{Li}_2(1-w) + 2\zeta_2\right)\Bigg) - \\
&- N(N-N_F)\frac{1}{12}\left(\frac{\ln(w)}{(1-w)}\right) + \left(\frac{N^2+1}{N^2}\right)\frac{3}{8}\left(\frac{\ln(w)}{(1-w)}\right). \qquad (4.25)
\end{aligned}
$$

## 5. The gluon-gluon splitting amplitudes

The gluon-gluon splitting amplitudes can be extracted from the Higgs decay into gluons in the limit where the top quark is very heavy and an effective Higgs-gluon-gluon vertex is induced. Labelling the gluon momenta alphabetically we write the unrenormalised amplitude in terms of the single colour ordered amplitude,

$$|\mathcal{M}_{H\to AB}\rangle = \boldsymbol{\delta}_{AB}\mathcal{A}_H(A,B)\,, \qquad (5.1)$$

where,

$$
\begin{aligned}
\mathcal{A}_H(A^+,B^+) &= A_H\,[AB]^2\,, &(5.2)\\
\mathcal{A}_H(A^+,B^-) &= 0\,. &(5.3)
\end{aligned}
$$

The amplitudes where the first gluon has negative helicity can be obtained by parity.

Similarly, we write the Higgs to three gluon amplitude as,

$$|\mathcal{M}_{H\to abc}\rangle = \boldsymbol{f}^{abc}\mathcal{A}_H(a,b,c)\,. \tag{5.4}$$

Again, there is only one colour ordering and the two independent helicity amplitudes for the three gluon decay are [85, 71],

$$\mathcal{A}_H(a^+,b^+,c^+) = -\alpha_H(y,z)\,\frac{1}{\sqrt{2}}\frac{s_{abc}^2}{\langle ab\rangle\langle bc\rangle\langle ac\rangle}\,, \tag{5.5}$$

$$\mathcal{A}_H(a^+,b^-,c^+) = \beta_H(y,z)\,\frac{1}{\sqrt{2}}\frac{[ac]^4}{[ab][bc][ca]}\,, \tag{5.6}$$

where $y = s_{ab}/s_{abc}$ and $z = s_{ac}/s_{abc}$.

The other helicity amplitudes are obtained by the usual parity and charge conjugation relations. For the present purposes, the only useful one is,

$$\mathcal{A}_H(a^-,b^-,c^+) = \beta_H(z,y)\,\frac{1}{\sqrt{2}}\frac{\langle ab\rangle^4}{\langle ab\rangle\langle bc\rangle\langle ca\rangle}\,. \tag{5.7}$$

The limit where gluons $a$ and $b$ are collinear corresponds to $y = s_{ab}/s_{abc} \to 0$. We assign a momentum fraction $(1-w)$ to particle $a$. In this limit, we see that,

$$\begin{aligned}
\mathcal{A}_H(a^+,b^+,c^+) &\xrightarrow{a||b} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{w(1-w)}\langle ab\rangle}\left(\alpha_H(y,z)\Big|_{y\to 0}\right)[AB]^2\,,\\
\mathcal{A}_H(a^+,b^-,c^+) &\xrightarrow{a||b} -\frac{1}{\sqrt{2}}\frac{(1-w)^2}{\sqrt{w(1-w)}[ab]}\left(\beta_H(y,z)\Big|_{y\to 0}\right)[AB]^2\,,\\
\mathcal{A}_H(a^-,b^-,c^+) &\xrightarrow{a||b} -\frac{1}{\sqrt{2}}\frac{\langle ab\rangle}{\sqrt{w(1-w)}[ab]^2}\left(\frac{s_{ab}^2}{s_{abc}^2}\beta_H(z,y)\Big|_{y\to 0}\right)[AB]^2\,.
\end{aligned} \tag{5.8}$$

Note that the terms in round brackets are evaluated in the collinear limit, $y \to 0$.

The two-particle helicity amplitude coefficient $A_H$ has the perturbative expansion,

$$A_H = C_{H\to GG}\left[A_H^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)A_H^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_H^{(2)} + \mathcal{O}(\alpha_s^3)\right]. \tag{5.9}$$

Similarly, the three-particle helicity amplitude coefficients $\alpha_H$, and $\beta_H$ are given by,

$$\Omega_H(y,z) = C_{H\to ggg}\left[\Omega_H^{(0)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)\Omega_H^{(1)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)^2 \Omega_H^{(2)}(y,z) + \mathcal{O}(\alpha_s^3)\right], \tag{5.10}$$

for $\Omega_H = \alpha_H, \beta_H$.

### 5.1 The tree-level gluon-gluon splitting amplitudes

At leading order

$$\alpha_H^{(0)}(y,z) = \beta_H^{(0)}(y,z) = 1 \qquad \text{and} \qquad A_H^{(0)} = 1\,, \tag{5.11}$$

and we can immediately read off the tree-level gluon-gluon splitting functions,

$$\begin{aligned}
\mathrm{Split}_{-}^{G\to gg,(0)}(a^+,b^+) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{w(1-w)}\ \langle ab\rangle}\,,\\
\mathrm{Split}_{-}^{G\to gg,(0)}(a^+,b^-) &= -\frac{1}{\sqrt{2}}\frac{(1-w)^2}{\sqrt{w(1-w)}\ [ab]}\,,\\
\mathrm{Split}_{-}^{G\to gg,(0)}(a^-,b^-) &= 0\,,
\end{aligned} \tag{5.12}$$

and by symmetry,

$$\mathrm{Split}_{-}^{G\to gg,(0)}(a^-,b^+) = -\frac{1}{\sqrt{2}}\frac{w^2}{\sqrt{w(1-w)}\ [ab]}\,. \tag{5.13}$$

The amplitudes where a gluon with negative helicity splits are obtained by parity,

$$\mathrm{Split}_{+}^{G\to gg,(0)}(a^{\lambda_a},b^{\lambda_b}) = \left(\mathrm{Split}_{-}^{G\to gg,(0)}(a^{-\lambda_a},b^{-\lambda_b})\right)^*\,, \tag{5.14}$$

and recalling that $[ab]^* = -\langle ab\rangle$.

## 5.2 The one-loop gluon-gluon splitting amplitudes

As in the quark-gluon case, the one loop splitting functions are presented in ref. [25], and we reproduce them here in the 't Hooft-Veltman regularisation scheme ($\delta_R = 1$) for the sake of completeness. For the amplitude that vanishes at tree-level we have,

$$\begin{aligned}
\mathrm{Split}_{-}^{G\to gg,(1)}(a^-,b^-) = c_\Gamma\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^\epsilon &\frac{w(1-w)}{(1-2\epsilon)(2-2\epsilon)(3-2\epsilon)}\,(N-N\epsilon-N_F)\times\\
&\times -\frac{1}{\sqrt{2}}\frac{\langle ab\rangle}{\sqrt{w(1-w)}[ab]^2}\,,
\end{aligned} \tag{5.15}$$

while the other splitting functions are,

$$\begin{aligned}
&r_S^{G\to gg,(1)}(\mp,a^\pm,b^\pm) =\\
&\quad= c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^\epsilon\left\{\frac{N}{2\epsilon^2}\left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon)\left(\frac{w}{1-w}\right)^\epsilon+\sum_{m=1}^\infty 2\epsilon^{2m-1}\mathrm{Li}_{2m-1}\left(\frac{1-w}{-w}\right)\right]+\right.\\
&\qquad\qquad\left.+\frac{w(1-w)}{(1-2\epsilon)(2-2\epsilon)(3-2\epsilon)}\,(N-N\epsilon-N_F)\right\},
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&r_S^{G\to gg,(1)}(\mp,a^\pm,c^\mp) = r_S^{G\to gg,(1)}(\mp,a^\mp,c^\pm) =\\
&\quad= c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^\epsilon\frac{N}{2\epsilon^2}\left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon)\left(\frac{w}{1-w}\right)^\epsilon+\sum_{m=1}^\infty 2\epsilon^{2m-1}\mathrm{Li}_{2m-1}\left(\frac{1-w}{-w}\right)\right].
\end{aligned} \tag{5.17}$$

Note that eqs. (5.16) and (5.17) are symmetric under $w \leftrightarrow (1-w)$. Note also that because of our different normalisations, the one-loop splitting function differs by a factor of $N/2$ compared to ref. [25]. We have checked that we reproduce these expressions through to $\mathcal{O}(\epsilon^2)$.

### 5.3 The two-loop gluon-gluon splitting amplitudes

Using explicit calculations for the unrenormalised two-loop coefficients for the $H \to gg$ and $H \to ggg$ decays, we find that the unrenormalised splitting function $r_S^{G\to gg,(2)}(\epsilon)$ for the helicity combinations that are non-vanishing at tree level have a pole structure determined by eq. (3.7) with finite remainders given by,

$$
\begin{aligned}
r_S^{G\to gg,(2),\mathrm{fin}}&(-,g^+,g^+) = \\
&= +N^2\left(-\frac{11}{32}\zeta_4+\frac{5}{24}\zeta_2-\frac{523}{432}\right)+\beta_0 N\Bigg(-\mathrm{Li}_3(w)-\mathrm{Li}_3(1-w)+\frac{1}{2}\mathrm{Li}_2(w)\ln(w)+ \\
&\qquad +\frac{1}{2}\mathrm{Li}_2(1-w)\ln(1-w)+\frac{79}{24}\zeta_3+ \\
&\qquad +\frac{25}{48}\zeta_2-\frac{65}{216}\Bigg)+ \\
&\quad + w(1-w)\Bigg(\frac{47}{108}N^2-\frac{23}{216}NN_F-\frac{11}{54}N_F^2-\frac{1}{8}\frac{N_F}{N}+\frac{N(N-N_F)}{12}\times \\
&\qquad \times\left(2\zeta_2+\frac{\ln(w)}{(1-w)}+\frac{\ln(1-w)}{w}\right)\Bigg)-w^2(1-w)^2(N-N_F)^2\frac{1}{72}\,, \qquad (5.18)
\end{aligned}
$$

$$
\begin{aligned}
r_S^{G\to gg,(2),\mathrm{fin}}&(-,g^+,g^-) = \\
&= +N^2\left(-\frac{3}{8}\ln(w)-\frac{3}{8}\ln(1-w)-\frac{11}{32}\zeta_4+\frac{5}{24}\zeta_2-\frac{523}{432}\right)+ \\
&\quad +\beta_0 N\Bigg(-\mathrm{Li}_3(w)+\frac{1}{2}\mathrm{Li}_2(w)\ln(w)+\frac{1}{4}\ln(w)+\frac{1}{2}\ln(1-w)\zeta_2+\frac{1}{4}\ln(1-w) \\
&\qquad +\frac{55}{24}\zeta_3+\frac{25}{48}\zeta_2-\frac{65}{216}\Bigg)- \\
&\quad -N\beta_0\frac{w}{(1-w)}\left(\mathrm{Li}_3(w)-\zeta_3-\frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w)-\zeta_2\right)\right)-N(N-N_F)\times \\
&\quad \times\Bigg(\frac{w(1+w)}{6(1-w)^3}\left(\mathrm{Li}_3(w)-\zeta_3-\frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w)-\zeta_2\right)\right)+ \\
&\qquad +\frac{w}{6(1-w)^2}\left(-\mathrm{Li}_2(1-w)+2\zeta_2\right)+\frac{1}{12}\left(\frac{\ln(w)}{(1-w)}\right)\Bigg). \qquad (5.19)
\end{aligned}
$$

Finally, for the amplitude that vanishes at tree-level we find,

$$
\begin{aligned}
\mathrm{Split}_-^{G\to gg,(2)}&(a^-,b^-) = \\
&= c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{ab}}\right)^\epsilon\left(\frac{N}{2\epsilon^2}(w(1-w))^{-\epsilon}-\frac{2\beta_0}{\epsilon}\right)\mathrm{Split}_-^{G\to gg,(1)}(a^-,b^-)-\frac{1}{\sqrt{2}}\frac{\langle ab\rangle}{\sqrt{w(1-w)}[ab]^2}\times \\
&\times\Bigg[N(N-N_F)\left(\frac{1}{6}\ln(1-w)+w\frac{1}{6}(\ln(w)-\ln(1-w))\right)-w(1-w)\times \\
&\quad \times\Bigg(\left(-\frac{101}{54}+\frac{\zeta_2}{12}-\frac{1}{6}\ln(w)\ln(1-w)\right)N^2+\left(\frac{451}{216}-\frac{\zeta_2}{12}+\frac{1}{6}\ln(w)\ln(1-w)\right)NN_F+ \\
&\qquad +\frac{1}{8}\frac{N_F}{N}-\frac{5}{54}N_F^2\Bigg)\Bigg]+\mathcal{O}(\epsilon)\,. \qquad (5.20)
\end{aligned}
$$

We have checked that the expressions presented here agree with those given in ref. [1].

## 6. The quark-antiquark splitting amplitudes

The quark-antiquark splitting amplitudes can be extracted from the Higgs decay into quarks and gluons in the large $m_t$ limit, which factorises onto the $H \to GG$ amplitudes in the collinear limit. The single colour ordered amplitude is obtained from the full amplitude by,

$$|\mathcal{M}_{H\to q\bar{q}g}\rangle = \boldsymbol{T}^c_{ab}\mathcal{A}_H(q,\bar{q},g)\,. \tag{6.1}$$

There is one independent helicity amplitude for the $H \to qg\bar{q}$ decay which is given by [85, 71],

$$\mathcal{A}_H(q^+,\bar{q}^-,g^+) = \gamma_H(y,z)\,\frac{1}{\sqrt{2}}\frac{[qg]^2}{[q\bar{q}]}\,, \tag{6.2}$$

where $y = s_{q\bar{q}}/s_{q\bar{q}g}$ and $z = s_{qg}/s_{q\bar{q}g}$. All other helicity amplitudes are either zero due to helicity conservation or can be related by charge conjugation and parity.

In the collinear quark and antiquark limit, $y = s_{q\bar{q}}/s_{q\bar{q}g} \to 0$. When the quark carries momentum fraction $(1-w))$ we see that,

$$\mathcal{A}_H(q^+,\bar{q}^-,g^+) \stackrel{q||\bar{q}}{\longrightarrow} \frac{1}{\sqrt{2}}\frac{w}{[q\bar{q}]}\left(\gamma_H(y,z)\Big|_{y\to 0}\right)[AB]^2\,. \tag{6.3}$$

Note that the terms in round brackets are evaluated in the collinear limit, $y \to 0$.

The three-particle helicity amplitude coefficient $\gamma_H$, has a perturbative expansion given by,

$$\gamma_H(y,z) = C_{H\to q\bar{q}g}\left[\gamma_H^{(0)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)\gamma_H^{(1)}(y,z) + \left(\frac{\alpha_s}{2\pi}\right)^2\gamma_H^{(2)}(y,z) + \mathcal{O}(\alpha_s^3)\right]. \tag{6.4}$$

### 6.1 The tree-level quark-antiquark splitting amplitudes

At leading order

$$\gamma_H^{(0)} = 1\,. \tag{6.5}$$

Together with the tree amplitude for $H \to GG$ in eq. (5.3), we can immediately read off the tree-level quark-antiquark splitting function,

$$\mathrm{Split}_-^{G\to q\bar{q},(0)}(q^+,\bar{q}^-) = \frac{1}{\sqrt{2}}\frac{w}{[q\bar{q}]}\,. \tag{6.6}$$

### 6.2 The one-loop quark-antiquark splitting amplitudes

As in the quark-gluon and gluon-gluon case, the one loop splitting functions are presented in ref. [25]. In the 't Hooft-Veltman regularisation scheme ($\delta_R = 1$),

$$\begin{aligned}
&r_S^{G\to q\bar{q},(1)}(-,q^+,\bar{q}^-;\epsilon) = \\
&= c_\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{-s_{q\bar{q}}}\right)^\epsilon\left\{\frac{N}{2\epsilon^2}\left[1-\Gamma(1-|!\epsilon)\Gamma(1+\epsilon)\left(\frac{w}{1-w}\right)^\epsilon + \sum_{m=1}^{\infty}2\epsilon^{2m-1}\mathrm{Li}_{2m-1}\left(\frac{1-w}{-w}\right)\right] + \right.\\
&\qquad + N\left(\frac{13}{12\epsilon(1-2\epsilon)} + \frac{1}{6(1-2\epsilon)(3-2\epsilon)}\right)\\
&\qquad + \frac{1}{N}\left(\frac{1}{2\epsilon^2} + \frac{3}{4\epsilon(1-2\epsilon)} + \frac{1}{2(1-2\epsilon)}\right) + \\
&\qquad \left. + N_F\left(-\frac{1}{3\epsilon(1-2\epsilon)} + \frac{1}{3(1-2\epsilon)(3-2\epsilon)}\right)\right\}.
\end{aligned} \tag{6.7}$$

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Note that because of our different normalisations, the one-loop splitting function differs by a factor of $N/2$ compared to ref. [25]. We have checked that we agree with this expression through to $\mathcal{O}(\epsilon^2)$.

### 6.3 The two-loop quark-antiquark splitting amplitudes

Using explicit calculations for the unrenormalised two-loop coefficients for the $H \to gg$ and $H \to q\bar{q}g$ processes, we find that the unrenormalised splitting function $r_S^{G\to q\bar{q},(2)}(\epsilon)$ has a pole structure determined by eq. (3.7) with a finite remainder given by,

$$
\begin{aligned}
&r_S^{G\to q\bar{q},(2),\mathrm{fin}}(-,q^+,\bar{q}^+) = \\
&= +N^2\bigg(\frac{5}{4}\mathrm{Li}_4(w) + \frac{5}{4}\mathrm{Li}_4(1-w) - \frac{1}{2}\mathrm{Li}_3(w)\ln(w) - \frac{3}{4}\mathrm{Li}_3(w)\ln(1-w) - \frac{3}{4}\mathrm{Li}_3(w) - \\
&\qquad - \frac{3}{4}\mathrm{Li}_3(1-w)\ln(w) - \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) - \frac{1}{4}\mathrm{Li}_2(w)^2 - \\
&\qquad - \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) + \frac{3}{8}\mathrm{Li}_2(w)\ln(w) + \frac{1}{4}\mathrm{Li}_2(w)\zeta_2 + \frac{1}{8}\mathrm{Li}_2(w) - \\
&\qquad - \frac{3}{8}\ln(w)^2\ln(1-w)^2 + \frac{3}{4}\ln(w)\ln(1-w)\zeta_2 + \frac{1}{8}\ln(w)\ln(1-w) + \frac{5}{4}\ln(w)\zeta_3 - \\
&\qquad - \frac{29}{48}\ln(w)\zeta_2 + \frac{121}{864}\ln(w) + \frac{5}{4}\ln(1-w)\zeta_3 - \frac{11}{48}\ln(1-w)\zeta_2 + \frac{121}{864}\ln(1-w) + \\
&\qquad + \frac{23}{16}\zeta_4 + \frac{5}{24}\zeta_3 - \frac{31}{96}\zeta_2 - \frac{14233}{3456}\bigg) + \\
&\quad + \bigg(\frac{5}{4}\mathrm{Li}_4(w) + \frac{5}{4}\mathrm{Li}_4(1-w) - \frac{1}{2}\mathrm{Li}_3(w)\ln(w) - \frac{3}{4}\mathrm{Li}_3(w)\ln(1-w) + \frac{3}{4}\mathrm{Li}_3(w) - \\
&\qquad - \frac{3}{4}\mathrm{Li}_3(1-w)\ln(w) - \frac{1}{2}\mathrm{Li}_3(1-w)\ln(1-w) - \frac{1}{4}\mathrm{Li}_2(w)^2 - \\
&\qquad - \frac{1}{4}\mathrm{Li}_2(w)\ln(w)\ln(1-w) - \frac{3}{8}\mathrm{Li}_2(w)\ln(w) + \frac{1}{4}\mathrm{Li}_2(w)\zeta_2 + \frac{1}{8}\mathrm{Li}_2(w) - \\
&\qquad - \frac{3}{8}\ln(w)^2\ln(1-w)^2 + \frac{3}{4}\ln(w)\ln(1-w)\zeta_2 + \frac{1}{8}\ln(w)\ln(1-w) - \frac{1}{4}\ln(w)\zeta_3 + \\
&\qquad + \frac{5}{16}\ln(w)\zeta_2 - \frac{41}{108}\ln(w) - \frac{1}{4}\ln(1-w)\zeta_3 - \frac{1}{16}\ln(1-w)\zeta_2 - \frac{41}{108}\ln(1-w) + \\
&\qquad + \frac{13}{32}\zeta_4 - \frac{749}{144}\zeta_3 + \frac{47}{48}\zeta_2 - \frac{11389}{2592}\bigg) + \\
&\quad + \frac{1}{N^2}\bigg(-\frac{3}{2}\ln(w)\zeta_3 + \frac{3}{4}\ln(w)\zeta_2 - \frac{3}{32}\ln(w) - \frac{3}{2}\ln(1-w)\zeta_3 + \frac{3}{4}\ln(1-w)\zeta_2 - \\
&\qquad - \frac{3}{32}\ln(1-w) - \frac{11}{8}\zeta_4 - \frac{15}{8}\zeta_3 + \frac{29}{16}\zeta_2 - \frac{1}{128}\bigg) + \\
&\quad + NN_F\bigg(\frac{1}{24}\ln(w)\zeta_2 + \frac{5}{8}\ln(w) + \frac{1}{24}\ln(1-w)\zeta_2 + \frac{5}{8}\ln(1-w) - \frac{5}{6}\zeta_3 + \\
&\qquad + \frac{31}{144}\zeta_2 + \frac{1717}{864}\bigg) + \\
&\quad + \frac{N_F}{N}\bigg(-\frac{1}{8}\ln(w)\zeta_2 + \frac{79}{216}\ln(w) - \frac{1}{8}\ln(1-w)\zeta_2 + \frac{79}{216}\ln(1-w) - \frac{53}{72}\zeta_3 -
\end{aligned}
$$

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$$
\begin{aligned}
&\qquad\qquad \left.\frac{7}{24}\zeta_2+\frac{5221}{2592}\right)+\\
&\quad +N_F^2\left(-\frac{5}{27}\ln(w)-\frac{5}{27}\ln(1-w)-\frac{1}{36}\zeta_2-\frac{1}{2}\right)+\\
&\quad +N^2\left(\frac{w(3w-2)}{4(1-w)^2}\left(\mathrm{Li}_3(w)-\zeta_3-\frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w)-\zeta_2\right)\right)+\right.\\
&\qquad\qquad \left.+\frac{w}{8(1-w)}\left(-\mathrm{Li}_2(1-w)+2\zeta_2\right)\right)\\
&+\left(\frac{w(4-3w)}{4(1-w)^2}\left(\mathrm{Li}_3(w)-\zeta_3-\frac{1}{2}\ln(w)\left(\mathrm{Li}_2(w)-\zeta_2\right)\right)+\right.\\
&\qquad \left.+\frac{w}{8(1-w)}\left(-\mathrm{Li}_2(1-w)+2\zeta_2\right)\right).
\end{aligned}
\tag{6.8}
$$

## 7. Limiting properties

It is useful consider various limiting cases of the splitting functions in order to get a more physical picture of what is going on.

In tables 7 and 7 we show the leading behaviour of the splitting functions as a polynomial in $w$ for $w \to 0$ and $(1-w)$ for $w \to 1$. $a_i, b_i, c_i$ and $d_i$ are the $i$-loop coefficients multiplying the leading term. Both limits correspond to the production of a soft particle. When the particle is a gluon, and the helicity of the hard particle is inherited from the parent, the tree splitting function is singular ($Q^+ \to q^+g^\pm$, $G^+ \to g^+g^\pm$ as $w \to 0$ and $G^+ \to g^+g^+$ as $w \to 1$). In each of these cases, the one- and two-loop splitting functions produce the universal constants $a_1$ and $a_2$, indicating that the soft limit is independent of the particle type and the helicity of the gluon. In the soft quark (antiquark) limit, there is no polynomial enhancement.

Furthermore, when the helicity of the hard particle is flipped (as in the $Q^+ \to q^+g^-$ and $G^+ \to g^+g^-$ splittings as $w \to 1$) there is a tree-level suppression factor of at least $(1-w)$. In the two-loop results presented in the previous sections, the finite contributions to the helicity flip processes $Q^+ \to q^+g^-$, $G^+ \to g^+g^-$ and $G^+ \to q^+\bar{q}^+$ splittings contain

| Process | Split$^{(0)}$ | $r^{(1)}$ | $r^{(2)}$ |
|---|---|---|---|
| $Q^+ \to q^+g^+$ | $\frac{1}{\sqrt{w}}$ | $a_1$ | $a_2$ |
| $Q^+ \to q^+g^-$ | $\frac{1}{\sqrt{w}}$ | $a_1$ | $a_2$ |
| $G^+ \to g^+g^+$ | $\frac{1}{\sqrt{w}}$ | $a_1$ | $a_2$ |
| $G^+ \to g^+g^-$ | $\frac{1}{\sqrt{w}}$ | $a_1$ | $a_2$ |
| $G^+ \to q^+\bar{q}^+$ | $w$ | $b_1$ | $b_2$ |

| Process | Split$^{(0)}$ | Split$^{(1)}$ | Split$^{(2)}$ |
|---|---|---|---|
| $G^+ \to g^-g^-$ | 0 | $\sqrt{w}$ | $\sqrt{w}$ |

**Table 1:** Leading behaviour of splitting functions as $w \to 0$

| Process | Split$^{(0)}$ | $r^{(1),\mathrm{fin}}$ | $r^{(2),\mathrm{fin}}$ |
|---|---|---|---|
| $Q^+ \to q^+g^+$ | 1 | $c_1$ | $c_2$ |
| $Q^+ \to q^+g^-$ | $1-w$ | $a_1$ | $d_2$ |
| $G^+ \to g^+g^+$ | $\frac{1}{\sqrt{1-w}}$ | $a_1$ | $a_2$ |
| $G^+ \to g^+g^-$ | $\frac{(1-w)^2}{\sqrt{1-w}}$ | $a_1$ | $\frac{1}{1-w}$ |
| $G^+ \to q^+\bar{q}^+$ | 1 | $b_1$ | $b_2$ |

| Process | Split$^{(0)}$ | Split$^{(1)}$ | Split$^{(2)}$ |
|---|---|---|---|
| $G^+ \to g^-g^-$ | 0 | $\sqrt{1-w}$ | $\sqrt{1-w}$ |

**Table 2:** Leading behaviour of splitting functions as $w \to 1$

apparent quadratic, cubic and quadratic singularities as $w \to 1$. In each case, the singularity is softened by the behaviour of the numerator, such that only the $G^+ \to g^+ g^-$ splitting has a residual singularity of $1/(1-w)$. This is of course annihilated by the tree-level helicity flip suppression factor of $(1-w)^{3/2}$.

All of these limits show that the soft behaviour of the two-loop splitting functions is well behaved and consistent with the expected behaviour.

## 8. The two-loop soft splitting function

In the limit that a gluon becomes soft, the colour ordered amplitudes factorise in a similar way to eqs. (3.1)–(3.3). At tree level we have the usual result,

$$|\mathcal{M}_n^{(0)}(a^{\lambda_a}, b^{\lambda_b}, c^{\lambda_c}, \ldots)\rangle \xrightarrow{b\to 0} \mathcal{S}^{(0)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(0)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle\,. \tag{8.1}$$

The tree level soft splitting amplitudes are given by,

$$\mathcal{S}^{(0)}(a, b^+, c) = \frac{\sqrt{2}\langle ac\rangle}{\langle ab\rangle\langle bc\rangle}, \qquad \mathcal{S}^{(0)}(a, b^-, c) = -\frac{\sqrt{2}[ac]}{[ab][bc]}. \tag{8.2}$$

The soft amplitudes are independent of the helicities and particle types of the neighboring legs, $a$ and $c$.

The factorisation of one-loop colour ordered amplitudes in the soft limit proceeds as,

$$\begin{aligned}|\mathcal{M}_n^{(1)}(a^{\lambda_a}, b^{\lambda_b}, c^{\lambda_c}, \ldots)\rangle \xrightarrow{b\to 0}\ & \mathcal{S}^{(0)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(1)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle + \\ & + \mathcal{S}^{(1)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(0)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle\,.\end{aligned} \tag{8.3}$$

The one-loop soft factor has been found, to all orders in $\epsilon$, to be [25, 26] where it is expressed in terms of the tree-level soft factor,

$$\mathcal{S}^{(1)}(a, b^\pm, c) = r_{\rm soft}^{(1)}(a, b^\pm, c)\mathcal{S}^{(0)}(a, b^\pm, c)\,, \tag{8.4}$$

where,

$$r_{soft}^{(1)}(a, b^\pm, c) = -\frac{c_\Gamma}{2\epsilon^2} N \left(\frac{4\pi\mu^2(-s_{ac})}{(-s_{ab})(-s_{bc})}\right)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon)\,. \tag{8.5}$$

Two-loop amplitudes have a similar soft limit,

$$\begin{aligned}|\mathcal{M}_n^{(2)}(a^{\lambda_a}, b^{\lambda_b}, c^{\lambda_c}, \ldots)\rangle \xrightarrow{b\to 0}\ & \mathcal{S}^{(0)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(2)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle + \\ & + \mathcal{S}^{(1)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(1)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle + \\ & + \mathcal{S}^{(2)}(a, b^{\lambda_b}, c)|\mathcal{M}_{n-1}^{(0)}(a^{\lambda_a}, c^{\lambda_c}, \ldots)\rangle\,.\end{aligned} \tag{8.6}$$

As in the one-loop case, it is convenient to factor out the overall singularity as the tree soft factor,

$$\mathcal{S}^{(2)}(a, b^\pm, c) = r_{soft}^{(2)}(a, b^\pm, c)\mathcal{S}^{(0)}(a, b^\pm, c)\,. \tag{8.7}$$

We can use the two-loop quark-gluon and gluon-gluon splitting amplitudes presented in sections 4 and 5 to recover the soft limit. Compared to the collinear limit defined by eq. (3.4), the soft limit is obtained by taking $w$ small. In terms of invariants, $w \to$

$s_{bc}/s_{ab}$. Expanding the explicit expressions for $r^{G\to gg,(2)}(-,g^+,g^+)$ , $r^{G\to gg,(2)}(-,g^+,g^-)$ , $r^{Q\to qg,(2)}(\mp,q^\pm,g^\pm)$ and $r^{Q\to qg,(2)}(\mp,q^\pm,g^\mp)$ in the $w\to 0$ limit, we find the universal two-loop soft splitting factor to be,

$$
\begin{aligned}
r^{(2)}_{soft}(a,b^\pm,c) = {}& c_\Gamma^2(\epsilon)\frac{\Gamma(1+\epsilon)^2\Gamma(1-\epsilon)^2}{8\epsilon^4}\left(\frac{4\pi\mu^2(-s_{ac})}{(-s_{ab})(-s_{bc})}\right)^{2\epsilon}\times \\
&\times\Bigg(N^2-N\beta_0\epsilon-NK\epsilon^2+\left[N^2\left(-\frac{193}{27}+\zeta_3\right)+NN_F\frac{19}{27}\right]\epsilon^3+ \\
&+\left[N^2\left(-\frac{1142}{81}+\frac{88}{3}\zeta_3+\frac{7}{180}\pi^4\right)+NN_F\left(\frac{65}{81}-\frac{16}{3}\zeta_3\right)\right]\epsilon^4+\mathcal{O}(\epsilon^5)\Bigg).
\end{aligned}
\tag{8.8}
$$

Note that although the soft limit has simple QED-like factorisation properties for colour ordered amplitudes, the soft limit of the colour dressed amplitudes is rather more complicated. Nevertheless, the procedure for colour dressing is straightforward [26, 1].

## 9. Conclusions

In this paper we have studied the universal behaviour of two-loop amplitudes in the limit where two massless external partons become collinear using explicit calculations of $\gamma^*\to q\bar{q}g$ and $H\to 3$ partons. While we work with two-loop amplitudes in the specific case, our results generalise to processes involving more external particles. Each of the amplitudes has a trivial colour structure, so our results naturally apply to colour stripped (or ordered) amplitudes.

We find that the pole structure of the unrenormalised two-loop splitting amplitudes is given by eq. (3.7). After renormalisation, this is in agreement with the pole structure predicted by Catani [63] for generic renormalised two-loop amplitudes. A similar structure has been found for the gluon splitting function in ref. [1].

The finite remainders for the two independent quark-gluon splitting functions are given in eqs. (4.24) and (4.25), while similar results for the three different helicities needed to describe gluon-gluon splitting are given in eqs. (5.18)–(5.20). In this latter case, we find complete agreement with the results obtained in ref. [1] using a more general approach involving the unitarity method of sewing together tree (and one-loop) amplitudes to directly construct the two-loop splitting functions. Finally, the finite two-loop remainder for the one independent quark-antiquark splitting function is given in eq. (6.8).

Our results are valid for the time-like splitting of colour ordered amplitudes in the 't Hooft-Veltman dimensional regularisation scheme. The procedures for extending their range of validity into the space-like region or dressing with colour are straightforward [1].

We have studied the limits where one of the split partons is soft and, although there are apparent additional polynomial singularities as $w\to 1$ in some of the splitting functions, find that the logarithms and polylogarithms multiplying the explicit poles conspire to reduce the degree of the singularity by two powers. The net result is that the soft splittings which induce a helicity flip on the hard parton remain suppressed.



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For the processes where the helicity of the hard parton is conserved, we find that the soft limit of the radiated soft gluon is universal and is independent of the helicity or particle type of the hard parton. This universal factor is the two-loop soft splitting function and we give an explicit expression for it in eq. (8.8).

The splitting amplitudes presented here are another tool in the armoury of perturbative QCD. They can serve as a check on calculations of two-loop QCD amplitudes with more external legs. Together with the triple collinear limits of one-loop integrals [62] and the quadruple collinear limits of tree amplitudes [61], they may also be useful in rederiving the recently calculated [55, 56] three-loop contributions to the Altarelli-Parisi kernels [60]. Finally, they will provide one of the ingredients necessary for NNNLO computations of jet cross sections.

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## A. Infrared factorisation of two-loop amplitudes

Catani [63] has shown how to organize the infrared pole structure of the two-loop amplitudes renormalized in the $\overline{\mathrm{MS}}$ scheme in terms of the tree and renormalized one-loop amplitudes. In particular, the infrared behaviour of the one-loop coefficients is given by

$$|\mathcal{M}_{\mathcal{P}}^{(1)}\rangle = \boldsymbol{I}_{\mathcal{P}}^{(1)}(\epsilon)\left|\mathcal{M}_{\mathcal{P}}^{(0)}\right\rangle + \left|\mathcal{M}_{\mathcal{P}}^{(1),\mathrm{finite}}\right\rangle, \tag{A.1}$$

where $|\mathcal{M}_{\mathcal{P}}^{(1)}\rangle$ is the em renormalised $i$-loop amplitude. Similarly, the two-loop singularity structure is

$$\left|\mathcal{M}_{\mathcal{P}}^{(2)}\right\rangle = \boldsymbol{I}_{\mathcal{P}}^{(2)}(\epsilon)\left|\mathcal{M}_{\mathcal{P}}^{(0)}\right\rangle + \boldsymbol{I}_{\mathcal{P}}^{(1)}(\epsilon)\left|\mathcal{M}_{\mathcal{P}}^{(1)}\right\rangle + \left|\mathcal{M}_{\mathcal{P}}^{(2),\mathrm{finite}}\right\rangle, \tag{A.2}$$

where the infrared singularity operators $\boldsymbol{I}_{\mathcal{P}}^{(1)}$ and $\boldsymbol{I}_{\mathcal{P}}^{(2)}$ are process but not helicity dependent. Each of these operators produces terms of $\mathcal{O}(\epsilon)$ and the division between finite and non-finite terms is to some extent a matter of choice. Adding a term of $\mathcal{O}(1)$ to $\boldsymbol{I}^{(1)}$ is automatically compensated for by a change in $\boldsymbol{I}^{(2)}$.

For the virtual photon initiated processes, (2.6) and (2.2),

$$\boldsymbol{I}_{\gamma^*\to Q\bar{Q}}^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\mathtt{S}_{AB} - \frac{1}{N}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\mathtt{S}_{AB}\right], \tag{A.3}$$

$$\boldsymbol{I}_{\gamma^*\to q\bar{q}g}^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{3}{4\epsilon}+\frac{\beta_0}{2N\epsilon}\right)(\mathtt{S}_{ac}+\mathtt{S}_{bc}) - \frac{1}{N}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\mathtt{S}_{ab}\right], \tag{A.4}$$

where

$$\mathtt{S}_{ij} = \left(-\frac{4\pi\mu^2}{s_{ij}}\right)^{\epsilon}. \tag{A.5}$$

Note that on expanding $\mathtt{S}_{ij}$, imaginary parts are generated, the sign of which is fixed by the small imaginary part $+i0$ of $s_{ij}$.

For the Higgs processes, (2.7), (2.3) and (2.4), we have

$$\boldsymbol{I}^{(1)}_{H\to GG}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[2N\left(\frac{1}{\epsilon^2}+\frac{\beta_0}{N\epsilon}\right)\mathtt{S}_{AB}\right], \tag{A.6}$$

$$\boldsymbol{I}^{(1)}_{H\to ggg}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{\beta_0}{N\epsilon}\right)(\mathtt{S}_{ac}+\mathtt{S}_{bc}+\mathtt{S}_{ac})\right], \tag{A.7}$$

$$\boldsymbol{I}^{(1)}_{H\to qg\bar{q}}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left[N\left(\frac{1}{\epsilon^2}+\frac{3}{4\epsilon}+\frac{\beta_0}{2N\epsilon}\right)(\mathtt{S}_{ab}+\mathtt{S}_{bc})-\frac{1}{N}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\mathtt{S}_{ac}\right] \tag{A.8}$$

The two-loop singularity operator $\boldsymbol{I}^{(2)}_{\mathcal{P}}$ is given by,

$$\begin{aligned}\boldsymbol{I}^{(2)}_{\mathcal{P}}(\epsilon) = \Bigg(&-\frac{1}{2}\boldsymbol{I}^{(1)}_{\mathcal{P}}(\epsilon)\boldsymbol{I}^{(1)}_{\mathcal{P}}(\epsilon)-\frac{\beta_0}{\epsilon}\boldsymbol{I}^{(1)}_{\mathcal{P}}(\epsilon)+\\ &+e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\boldsymbol{I}^{(1)}_{\mathcal{P}}(2\epsilon)+\boldsymbol{H}^{(2)}_{\mathcal{P}}(\epsilon)\Bigg).\end{aligned} \tag{A.9}$$

In principle, $\boldsymbol{H}^{(2)}_{\mathcal{P}}$ contains colour correlations between the final state particles. However, for the processes at hand, these correlations vanish and each external coloured leg in the partonic process contributes independently to $\boldsymbol{H}^{(2)}_{\mathcal{P}}$

$$\boldsymbol{H}^{(2)}_{\mathcal{P}} = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)}\left(n_q H^{(2)}_q + n_g H^{(2)}_g\right), \tag{A.10}$$

where $n_g$ is the number of external gluons, $n_q$ is the number of external quarks and antiquarks for the process $\mathcal{P}$ and $H_q$ and $H_g$ are defined in eqs. (3.10) and (3.11) respectively.

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